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The Kac Equation with a Thermostatted Force Field

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We consider the Kac equation with a thermostatted force field and prove the existence of a global in time solution that converges weakly to a stationary state. As there is no an obvious candidate for the entropy functional, in this case, the convergence result is obtained via Fourier transform techniques.

KEY WORDS: Boltzmann equation, Kac equation, Gaussian thermostat.

1. INTRODUCTION

We consider the Kac equation with a particular kind of force field that conserves energy:

$$\frac{\partial}{\partial t} f + E \frac{\partial}{\partial v}((1 - \zeta(t)\, v)\, f) = Q(f, f), \tag{1}$$

where

$$\zeta(t) \;=\; \int_{\mathbb{R}} v\, f(v, t)\, dv. \tag{2}$$

In the right hand side, Q is the collision term

$$Q(f, f)(v, t) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left[f(v', t) f(v'_*, t) - f(v, t) f(v_*, t) \right] \frac{d\theta}{2\pi} dv_*. \tag{3}$$

This equation is derived as the limit when $N \to \infty$ of systems of N particles with one-dimensional velocities.[(1)] The particle positions are not taken into account, and just as in the original Kac model, and collisions between two particles with velocities v and v_* are modeled as rotations in the (v, v_*)-plane (see Eq. (6)). The term $E \frac{\partial}{\partial v}((1 - \zeta(t)\, v)\, f)$ in (1) which is new with respect to the original

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work of Kac, comes from adding a force field which accelerates the particles in the intervals between collisions. A simple model would be to accelerate each individual particle in the N-particle system with the same constant force,

$$\frac{dv_j}{dt} = E, \qquad (j = 1, \ldots, N),$$

but then clearly energy would continuously be added to the system, and to compensate that, the field is modified by a so-called thermostat in the following way. Denoting the N-dimensional *master vector* $\mathbf{V} = (v_1, \ldots, v_N)$, the accelerating *master field* is

$$\mathbf{E} - \frac{\mathbf{E} \cdot \mathbf{V}}{|\mathbf{V}|^2}\mathbf{V}, \tag{4}$$

where $\mathbf{E} = E(1, 1, \ldots, 1) \in \mathbb{R}^N$. To apply the thermostat amounts to projecting the constant master field onto the surface of constant energy, $|\mathbf{V}| = \mathit{const}$.

Thermostatted force fields have been introduced in statistical physics and in particular in molecular dynamics, because they are useful when studying non equilibrium stationary states.[2] In a previous paper,[3] we proved existens of stationary solutions to Eq (1). Depending on E, the strength of the field, these stationary states may have a singularity, but they always have a density.

In this paper we prove existence and uniqueness of a solution to the time dependent problem, and we also prove that the solutions converge to a stationary state. The existence result is based on integration along characteristics and successive approximations, and is rather straight forward. The proof of convergence to the stationary state is complicated by the lack of a natural entropy. Instead we prove that the Fourier transform of the solutions to (1) converges point-wise to the Fourier transform of the stationary solution.

The Kac equation is one of the simplest possible models of the Boltzmann equation, and it does not share all the physically relevant properties. In particular, momentum is not preserved by the collisions, and this changes the long time behavior. In fact, the Boltzmann equation with a thermostatted force field corresponding to (1) would be

$$\frac{\partial}{\partial t} f + E\nabla \cdot ((1 - \zeta_1(t)\mathbf{v}) f) = Q(f, f), \tag{5}$$

where in this case $\mathbf{v} \in \mathbb{R}^3$, and Q is the usual collision operator for the Boltzmann equation. Here $\zeta_1(t)$ is the first component of $\boldsymbol{\zeta}(t) = \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{v}, t)\, d\mathbf{v} \in \mathbb{R}^3$. This corresponds to the case where each particle is accelerated by a field of type $(E, 0, 0)$ before the thermostat is applied. In this case the collision operator does not contribute to the evolution of $\boldsymbol{\zeta}(t)$, which converges exponentially fast to $(1, 0, 0)$. It follows that $f(\mathbf{v}, t) \to \delta_{(1,0,0)}$ when $t \to \infty$, and so the stationary states are trivial. They are also in equilibrium, because the Dirac mass is a degenerated Maxwellian.

Indeed, the nonlinear Boltzmann equation may not be the most natural place to introduce thermostatted fields. We first learned about thermostats in the context of the Lorentz gas,[(4)] and seen from that point of view, it is rather be the linear Boltzmann equation that is relevant. However, if for example a linear collision term were added, then one would again obtain a system with non trivial stationary states.

A different type of modified Boltzmann equations, which have attracted much interest recently, and which also lead to non equilibrium situations, are models for granular media. There the collisions are dissipative, and in order to find non trivial stationary states, either the equations are rescaled according to the actual energy, or energy is added by putting the particles in a thermal bath. Such models are described e.g. in.[(5–7)]

This paper is organized as follows. First, in Sec 2 we give a formal derivation of the force field in the Kac equation. This is formal because we *assume* the propagation of chaos. Then in Sec 3, the existence and uniqueness theorem is stated and proven. This section is essentially a condensed form of the second author's doctoral thesis.[(8)] In Sec 4 we study the Fourier transform of eq. (1), and use this to prove that the solutions converge to a stationary state when $t \to \infty$. Finally, Sect 5 contains some numerical calculations to illustrate the theoretical results.

2. THE THERMOSTATTED KAC EQUATION

In,[(1)] Kac derived a nonlinear Boltzmann equation as the limit, when N goes to infinity, of a stochastic N-particle system. He considered a spatially homogeneous gas consisting of N point-particles with one-dimensional velocities $v_j \in \mathbb{R}$, $j = 1, 2, \ldots, N$. The time evolution of the gas forms a stochastic process, where at exponentially distributed time intervals, binary collisions take place as follows: a pair of velocities, say v_i and v_j, are selected randomly, and are assigned new velocities, v_i' and v_j', by a random rotation in $\mathbb{R}^2$, namely

$$\begin{aligned} v_i' &= v_i \cos\theta - v_j \sin\theta \\ v_j' &= v_i \sin\theta + v_j \cos\theta, \end{aligned} \tag{6}$$

where the scattering angle, θ, is chosen from a uniform distribution over $[-\pi, \pi)$. Distributions favoring some collisions over others have also been considered.[(9)] Clearly $v_i^2 + v_j^2 = {v_i'}^2 + {v_j'}^2$, and thus the total energy, $\sum_{i=1}^N v_i^2$, is conserved in the process. Hence, the Kac model defines a jump process on the $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$, which is normalized to have radius $\sqrt{N}$. For this process to mimic a system of real particles, one takes the collision frequency, i.e. the rate in the exponential distribution, to be proportional to N.

Let Ψ_N be the probability density of points in phase space each of which evolve according to the above Kac process. Then Ψ_N satisfies the master equation

$$\frac{\partial}{\partial t}\Psi_N(\mathbf{V}, t) = \mathrm{K}(\Psi_N)(\mathbf{V}, t), \tag{7}$$

where $\mathbf{V} = (v_1, \ldots, v_N) \in \mathbb{S}^{N-1}(\sqrt{N})$. K is the linear operator given by

$$\mathrm{K}(\Psi_N) = N(\tilde{\mathrm{K}} - \mathrm{I})(\Psi_N), \tag{8}$$

where I denotes the identity operator, and $\tilde{\mathrm{K}}$ is defined by

$$\tilde{\mathrm{K}}(\psi_N)(\mathbf{V}, t) = \binom{N}{2}^{-1} \sum_{1\le i<j\le N} \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi_N(A_{ij}(\theta)\mathbf{V}, t)\, d\theta,$$

with $A_{ij}(\theta)\mathbf{V} = (v_1, \ldots, v_i', \ldots, v_j', \ldots, v_N)$.

Let $k \le N$ be fixed. Define the k-particle marginal f_k^N by

$$f_k^N(v_1, \ldots, v_k, t) = \int_{\Omega_k} \Psi_N(v_1, \ldots, v_N, t)\, d\sigma_k, \tag{9}$$

where $\Omega_k = \mathbb{S}^{\mathrm{N}-1-k}(\sqrt{\mathrm{N} - \sum_{i=1}^k v_i^2})$ and $d\sigma_k$ denotes the normalized surface measure on Ω_k. The family $\{\Psi_N\}_{N=1}^{\infty}$ of probability densities is said to have the Boltzmann property, if

$$\lim_{N\to\infty} f_k^N(v_1, \ldots, v_k, t) = \prod_{j=1}^{k} \lim_{N\to\infty} f_1^N(v_j, t).$$

The main result in[1] is that, under suitable conditions on the initial data, the one particle marginal, f_1^N, converges, as $N \to \infty$, to a solution f of the Kac equation,

$$\frac{\partial}{\partial t} f(v, t) \;=\; Q(f, f)(v, t), \tag{10}$$

where the collision term, Q, has the form (3).

In this paper Kac's original stochastic model is modified by incorporating an external uniform force field which accelerates the particles between the collisions. This field acts equally on each particle, and in order to keep the total energy constant, it is projected onto the tangent plane to the energy surface – $\mathbb{S}^{\mathrm{N}-1}(\sqrt{\mathrm{N}})$. More precisely, between collisions, $\mathbf{V}$ evolves according to

$$\frac{d\mathbf{V}}{dt} = \mathbf{E} - \frac{\mathbf{E}\cdot\mathbf{V}}{|\mathbf{V}|^2}\mathbf{V}, \tag{11}$$

where $\mathbf{E} = \mathrm{E}(1, \ldots, 1) \in \mathbb{R}^N$. Eq. (11) can also be written component-wise as

$$\frac{dv_i}{dt} = \mathrm{E}\left(1 - \frac{\mathrm{J}}{\mathrm{U}} v_i\right), \quad i = 1, \ldots, N,$$

where $\boldsymbol{J} = \frac{1}{N} \sum_{i=1}^{N} v_i$ and $\mathrm{U} = \frac{1}{\mathrm{N}} \sum_{i=1}^{N} v_i^2$.

Let $\Psi_N(\mathbf{V}, 0)$ be an initial probability distribution of points on $\mathbb{S}^{N-1}(\sqrt{N})$. If each of these points evolve according to the modified Kac process, then $\Psi_N(\mathbf{V}, t)$ satisfies the master equation

$$\frac{\partial}{\partial t} \Psi_N + \nabla \cdot (\mathbf{F}\, \Psi_N) = \mathrm{K}(\Psi_N), \tag{12}$$

where K is as in (8), and $\mathbf{F}$ is the thermostatted force field given by

$$\mathbf{F} = \mathbf{E} - \frac{\mathbf{E} \cdot \mathbf{V}}{|\mathbf{V}|^2} \mathbf{V} = \mathrm{E}\left(1 - \frac{\mathrm{J}}{\mathrm{U}} v_1, \ldots, 1 - \frac{\mathrm{J}}{\mathrm{U}} v_N\right).$$

Assume that the family $\{\Psi_{\mathrm{N}}\}_{\mathrm{N}=1}^{\infty}$ form a sequence of C^1-solutions to (12). We differentiate f_1^{N} with respect to t, use (12), and rearrange the resulting terms to get

$$\frac{\partial}{\partial t} f_1^N(v_1, t) + \int_{\Omega_1} \nabla \cdot (\mathbf{F}\, \Psi_N)\, d\sigma_1 = \int_{\Omega_1} \mathrm{K}(\Psi_N)\, d\sigma_1. \tag{13}$$

If the densities Ψ_N are symmetric with respect to permutation of the arguments, and if the family $\{\Psi_N\}_{N=1}^{\infty}$ satisfies the Boltzmann property for all t, then it is not difficult to show that the right hand side of (13) converges to the collision operator (3), when $N \to \infty$. The difficulty lies in proving that the Boltzmann property propagates in time. One says then that "chaos propagates," and Kac proved that this is true in the case of the Kac model. In this paper we *assume* that the propagation of chaos still holds with a force term, and hence that the right-hand side does not change by the addition of a force term.

The second term on the left-hand side of (13) is studied in,[8] where it is shown that under the assumption of propagation of chaos, and some additional regularity condition,

$$\int_{\Omega_1} \nabla \cdot (\mathbf{F}\, \Psi_N)\, d\sigma_1 \to \mathrm{E} \frac{\partial}{\partial v_1}((1 - \zeta(t) v_1) f),$$

when $N \to \infty$. The function $\zeta(t)$ is given by (2), and it is tacitly assumed that U, the mean energy of a particle, is equal to one. The main effort is made in proving that

$$\int_{\Omega_1} \left(1 - \frac{\mathrm{J}}{\mathrm{U}} v_1\right) \Psi_N\, d\sigma_1 \to (1 - \zeta(t)\, v_1) f,$$

when $N \to \infty$. This then shows, at least formally, that, if $f(v,t) = \lim_{N\to\infty} f_1^N(v,t)$ exists, then f satisfies (1), which constitutes the thermostatted Kac equation. To make the proof rigorous, one needs to prove that propagation of chaos holds.

3. EXISTENCE OF SOLUTION

In this section we study the initial value problem to Eq. (1). The main result of this section is

Theorem 1. *Let $f_0 \geq 0$ with $\int_{\mathbb{R}} f_0(v)\,dv = 1$ be given. Then there exists a non-negative $f \in C((0,\infty); L^1(\mathbb{R}))$, which is a mild solution to (1)-(2) with $f(v,0) = f_0(v)$, and such that $\int_{\mathbb{R}} f(v,t)\,dv = \int_{\mathbb{R}} v^2 f(v,t)\,dv = 1$.*

Remark 1. By a *mild* solution we mean a solution to the integral equation that is obtained by integration along characteristics, and here we actually mean *exponential mild* solution. The exact form of the integrated equation can be seen in Eq. (27) and (28) below. This notion of solutions is weaker than that of strong solutions, where all terms in (1) are assumed be well defined in L^1.

We first state and prove a result concerning the function $\zeta(t)$.

Lemma 1. *Let $f_0 \geq 0$ with $\int_{\mathbb{R}} f_0(v)\,dv = 1$ be given. Assume that there exists $f \geq 0$, solution to (1)-(2) such that $\int_{\mathbb{R}} f(v,t)\,dv = 1$, and $|v| f(v,t) \to 0$ as $|v| \to \infty$, for each $t \geq 0$. Then $\zeta(t)$ satisfies*

$$\frac{d}{dt}\zeta(t) = E\,(1 - \zeta(t)^2) - \zeta(t), \tag{14}$$

$$\zeta(0) = \zeta_0 \equiv \int_{\mathbb{R}} v\, f_0(v)\,dv. \tag{15}$$

This can be solved explicitly as

$$\zeta(t) = \frac{\zeta_+(\zeta_- - \zeta_0) - \zeta_-(\zeta_+ - \zeta_0)e^{-\sqrt{1+4E^2}\,t}}{(\zeta_- - \zeta_0) - (\zeta_+ - \zeta_0)e^{-\sqrt{1+4E^2}\,t}}, \tag{16}$$

where

$$\zeta_\pm = \frac{2E}{1 \pm \sqrt{1+4E^2}}. \tag{17}$$

Proof. Since $\int_{\mathbb{R}} f(v,t)\,dv = 1$, the collision term can be written as $Q(f,f) = Q_+(f,f) - f$, where

$$Q_+(f,f)(v,t) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} f(v',t) f(v'_*,t) \frac{1}{2\pi}\, d\theta\, dv_*.$$

Because $\int_{\mathbb{R}} v\, Q_+(f,f)\,dv = 0$, we get

$$\int_{\mathbb{R}} v\, Q(f,f)\,dv \;=\; -\zeta(t). \tag{18}$$

From the definition of $\zeta(t)$ it also follows that

$$\frac{d}{dt}\zeta(t) = \int_{\mathbb{R}} v \frac{\partial}{\partial t} f(v,t)\,dv.$$

Using (1) and doing a partial integration directly leads to Eq. (14)–(15) which has the unique solution (16). ∎

Next we study the initial value problem

$$\frac{\partial}{\partial t} f \;+\; E\,\frac{\partial}{\partial v}((1 - \bar{\zeta}(t)\,v)f) \;=\; Q(f,f), \quad (t > 0), \tag{19}$$

$$f(v,0) = f_0(v)\,, \tag{20}$$

where $\bar{\zeta}(t)$ is given by (16). This is exactly like (1)–(2) except that $\zeta(t) = \int_{\mathbb{R}} v\, f(v,t) dv$ is replaced by the known function $\bar{\zeta}(t)$.

First we rewrite (19) as

$$\frac{\partial}{\partial t} f \;+\; E(1 - \bar{\zeta}(t)\,v)\frac{\partial}{\partial v} f + (1 - E\,\bar{\zeta}(t)\,v) f \;=\; Q_+(f,f). \tag{21}$$

This constitutes a first-order semi-linear equation that can be solved by integration along characteristics and iterative procedure as used by Arkeryd.[10] After integration along characteristics (21) becomes

$$\frac{d}{dt} f^{\#} + (1 - E\,\bar{\zeta}(t)) f^{\#} \;=\; Q_+(f,f)^{\#}, \tag{22}$$

where, for each $v \in \mathbb{R}$, we use the notations

$$f^{\#}(v,t) = f(V(v,t),t), \tag{23}$$

$$Q_+(f,f)^{\#}(v,t) = Q_+(f,f)(V(v,t),t). \tag{24}$$

Here we make use of the transformation

$$V(v,t) = \psi_t(v) \equiv v\, e^{-\lambda(t)} + E\, e^{-\lambda(t)} \int_0^t e^{\lambda(s)}\, ds, \tag{25}$$

where $\lambda(t) = E \int_0^t \bar{\zeta}(s)ds$. The Jacobian of the transformation in (25) is $J = e^{-\lambda(t)} > 0$. Moreover,

$$\psi_t^{-1}(V) \;=\; V\, e^{\lambda(t)} \;-\; E \int_0^t e^{\lambda(s)}\, ds. \tag{26}$$

For notational convenience, we write $\Lambda(t) = \int_0^t \left(1 - E\,\bar{\zeta}(s)\right) ds$ or, in other words $\Lambda(t) = t - \lambda(t)$.

Let $T > 0$ be fixed. The integral form of (22) in the time interval $[0, T]$ is

$$f^{\#}(v,t) = e^{-\Lambda(t)} f_0^{\#}(v,0) + e^{-\Lambda(t)} \int_0^t e^{\Lambda(\tau)} Q_+(f,f)^{\#}(v,\tau) d\tau. \tag{27}$$

Using (26), and the notations in (23) and (24), we can rewrite (27) in terms of f

$$f(v,t) = \Gamma_{f_0}(f)(v,t), \tag{28}$$

where

$$\Gamma_{f_0}(f)(v,t) = e^{-\Lambda(t)} f_0(\psi_t^{-1}(v)) \\ + e^{-\Lambda(t)} \int_0^t e^{\Lambda(\tau)} Q_+(f,f)(\psi_\tau \circ \psi_t^{-1}(v), \tau)\, d\tau. \tag{29}$$

Theorem 1 is an immediate consequence of the following proposition.

Proposition 1. *Let $f_0 \geq 0$ with $\int_{\mathbb{R}} f_0(v)\,dv = 1$, and $\int_{\mathbb{R}} |v|^3 f_0(v)\,dv < \infty$ be given. Then there exists a unique, non-negative $f \in C((0,\infty); L^1(\mathbb{R}))$, which is a mild solution to (19)-(20). This solution satisfies $\int v\, f(v,t)\,dv = \bar{\zeta}(t)$, in addition to mass and energy being one.*

Remark 2. The condition that $\int_{\mathbb{R}} |v|^3 f_0(v)\,dv$ be bounded could surely be relaxed, but we keep it here for convenience.

Proof. We define two different sequences of approximations, based on Eq. (28), $\{f^{(n)}(\cdot,t)\}_{n=1}^{\infty}$ and $\{g^{(n)}(\cdot,t)\}_{n=1}^{\infty}$. The first one is obtained as

$$f^{(1)}(v,t) = 0, \tag{30}$$

$$f^{(n)}(v,t) = \Gamma_{f_0}(f^{(n-1)})(v,t), \quad (n > 1) \tag{31}$$

and has the merit of being monotonous, from which one may conclude that there is a limit $f \in L^1$, and that this limit is non-negative. The second sequence is defined in the same way, except that

$$g^{(1)}(v,t) = f_0(v).$$

This sequence is not necessarily monotonous, but has the advantage that moments up to the order two can be computed explicitly (and have the desired values), and that higher order moments may be easily estimated. In fact we claim that for each $t \geq 0$, and for each n

$$\int_{\mathbb{R}} g^{(n)}(v,t)\,dv = 1, \tag{32}$$

$$\int_{\mathbb{R}} v\,g^{(n)}(v,t)\,dv = \bar{\zeta}(t), \tag{33}$$

$$\int_{\mathbb{R}} v^2\,g^{(n)}(v,t)\,dv = 1. \tag{34}$$

where $\bar{\zeta}$ is given in (16), and moreover that all higher moments that initially are bounded remain bounded.

To prove the proposition, it suffices to show that $g^{(n)} \to f$ in L^1 which, in view of the boundedness of $\int_{\mathbb{R}} v^2\,g^{(n)}(v,t)\,dv$, would imply that $\int_{\mathbb{R}} v\,f(v,t)\,dv = \bar{\zeta}(t)$.

We recall the following easy relations (the first one being a direct consequence of the bilinearity of Q_+)

$$Q_+(f,f) - Q_+(g,g) = Q_+(f-g,f) + Q_+(g,f-g), \tag{35}$$

$$\int_{\mathbb{R}} Q_+(g,g)(w,\tau)\,dw = \left(\int_{\mathbb{R}} g(w,\tau)\,dw\right)^2, \tag{36}$$

$$\int_{\mathbb{R}} w\;Q_+(g,g)(w,\tau)\,dw = 0, \tag{37}$$

$$\int_{\mathbb{R}} w^2\;Q_+(g,g)(w,\tau)\,dw = \int_{\mathbb{R}} g(w,\tau)\,dw \int_{\mathbb{R}} w^2\,g(w,\tau)\,dw. \tag{38}$$

Using (29), (35), and the positivity of Q_+, it follows immediately that if $f^n \leq g^n$, then $f^{n+1} \leq g^{n+1}$. But $f_1 \leq g_1$, and so $f^n \leq g^n$ holds for all n (this is detailed below). We thus have

$$\int_{\mathbb{R}} \left|f - g^{(n)}\right| dv = 2\int_{\{f \geq g^{(n)}\}} \left(f - g^{(n)}\right) dv - \int_{\mathbb{R}} \left(f - g^{(n)}\right) dv.$$

But $\int_{\mathbb{R}} f(v,t)\,dv = \int_{\mathbb{R}} g^{(n)}(v,t)\,dv = 1$, so we find

$$\begin{aligned}\int_{\mathbb{R}} \left|f - g^{(n)}\right| dv &= 2\int_{\{f \geq g^{(n)}\}} \left(f - g^{(n)}\right) dv \\ &\leq 2\int_{\mathbb{R}} \left(f - f^{(n)}\right) dv \to 0,\end{aligned}$$

when $n \to \infty$. From the remark above, we know that $\int_{\mathbb{R}} |v|^3 g^{(n)} dv$ and from that we may conclude also that $\int_{\mathbb{R}} v^2 f(v,t)\,dv = 1$.

The same argument implies the uniqueness result in Theorem (1). In fact, if $f_*(v,t)$ is any solution to (28) then

$$f_*(v,t) = \Gamma_{f_0}(f_*)(v,t).$$

The monotonicity of Q_+, and thus that of Γ_{f_0}, then implies that, for all n,

$$f^{(n)}(v,t) \le \; f_*(v,t),$$

and the same holds for $f(v,t) = \lim_{n\to\infty} f^{(n)}(v,t)$. But

$$\int_{\mathbb{R}} f_*(v,t)dv = \int_{\mathbb{R}} f(v,t)dv = 1,$$

and thus $f_* = f$.

To conclude the proof of the theorem we need to prove that the sequences $\{f^{(n)}(\cdot,t)\}_{n=1}^{\infty}$ and $\{g^{(n)}(\cdot,t)\}_{n=1}^{\infty}$ behave as stated. We begin by the first one of these.

From the monotonicity of Q_+ and the assumption that $f_0 \ge 0$ we deduce that the iterates are all non-negative: It is immediate that

$$f^{(2)}(v,t) = e^{-\Lambda(t)} f_0(\psi_t^{-1}(v)), \tag{39}$$

from which we see $f^{(2)}(v,t) \ge f^{(1)}(v,t)$ for all $v \in \mathbb{R}$. Suppose now, for some $n \ge 3$, that $f^{(n-1)} \ge f^{(n-2)}$. The result now follows by induction and the formula

$$\begin{aligned} e^{-\Lambda(t)} \int_0^t e^{\Lambda(\tau)} \big[Q_+\big(f^{(n-1)} - f^{(n-2)}, f^{(n-1)}\big) \\ + Q_+\big(f^{(n-2)}, f^{(n-1)} - f^{(n-2)}\big)\big]\, d\tau. \end{aligned} \tag{40}$$

Since $\int_{\mathbb{R}} f_0(v)\,dv = 1$, the result in (39) gives $\int_{\mathbb{R}} f^{(2)}(v,t)\,dv = e^{-t} \le 1$. Suppose now, for some $n \ge 3$, that $\int_{\mathbb{R}} f^{(n-1)}(v,t)\,dv \le 1$. Then

$$\int_{\mathbb{R}} f^{(n)}(v,t)\,dv = e^{-t} + e^{-t} \int_0^t e^{\tau} \int_{\mathbb{R}} Q_+\big(f^{(n-1)}, f^{(n-1)}\big)(z,\tau)\,dz\,d\tau.$$

Using (36) and writing $M^{(n)}(t) = \int_{\mathbb{R}} f^{(n)}(v,t)dv$, we find that

$$M^{(n)}(t) = e^{-t} \; + \; e^{-t} \int_0^t e^{\tau} M^{(n-1)}(\tau)^2 d\tau.$$

Using the induction hypothesis gives

$$M^{(n)}(t) = \int_{\mathbb{R}} f^{(n)}(v,t)\,dv \;\le\; e^{-t} + e^{-t} \int_0^t e^{\tau} d\tau = 1.$$

The bounded, monotonically increasing sequence $\{f^{(n)}(\cdot,t)\}_{n=1}^{\infty}$ of non-negative terms has a non-negative limit $f(\cdot,t)$ in $L^1(\mathbb{R})$ such that $f^{(n)} \to f$ as $n \to \infty$.

Writing $M(t) = \int_{\mathbb{R}} f(v,t)dv$, we find that

$$M(t) = e^{-t} + e^{-t} \int_0^t e^{\tau} M(\tau)^2 d\tau,$$

which has the unique solution $M(t) \equiv 1$. From this we conclude, by Levi's theorem that $\int_{\mathbb{R}} f(v,t)\,dv = 1$. This says that f solves (28), as stated.

Next we turn to show that the sequence $\{g^{(n)}(\cdot,t)\}_{n=1}^{\infty}$ satisfies (32), (33), and (34). The assumption on f_0 gives that $\int_{\mathbb{R}} g^{(1)}(v,t)\,dv = 1$. We prove, by induction, that the same holds for all n. Suppose, for some $n \geq 2$, that $\int_{\mathbb{R}} g^{(n-1)}(v,t)\,dv = 1$. Integrating (31) with f_n replaced by g_n, and using (25) and (35), gives

$$e^t \int_{\mathbb{R}} g^{(n)}(v,t)\,dv = \int_{\mathbb{R}} f_0(v)\,dv \; + \int_0^t e^{\tau}\,d\tau.$$

The right hand side sums up to be e^t. With this we conclude, for all $t \geq 0$, and by induction for all n, that

$$\int_{\mathbb{R}} g^{(n)}(v,t)\,dv = 1.$$

To prove (33), we multiply both sides of (31) (with f replaced by g) by v, and integrate the result

$$\int_{\mathbb{R}} v\,g^{(n)}(v,t)\,dv = e^{-\Lambda(t)} \int_{\mathbb{R}} v f_0(\psi_t^{-1}(v))\,dv$$

$$+ e^{-\Lambda(t)} \int_{\mathbb{R}} \int_0^t e^{\Lambda(\tau)} v Q_+(g^{(n-1)}, g^{(n-1)})(\psi_\tau \circ \psi_t^{-1}(v), \tau)\,d\tau dv. \quad (41)$$

Use of (26) makes the first term in the above sum to be

$$e^{-t-\lambda(t)}(\zeta_0 + E\,q(0,t)), \quad (42)$$

where, for $\tau \leq t,\; q(\tau,t) = \int_\tau^t e^{\lambda(s)}\,ds$.

Using (25) and (26) once more, the second term in the sum (41) equals

$$e^{-t-\lambda(t)} \int_{\mathbb{R}} \int_0^t e^{\tau} \left[v e^{\lambda(\tau)} + Eq(\tau,t) \right] Q_+(g^{(n-1)}, g^{(n-1)})(v,\tau)\,d\tau dv.$$

Because $\int_{\mathbb{R}} w\; Q_+(g,g)(w,\tau)\,dw = 0$, this reduces to

$$E\,e^{-t-\lambda(t)} \int_0^t e^{\tau}\,q(\tau,t)\,d\tau. \quad (43)$$

Combining (42) and (43) makes the sum in (41) to take the form

$$e^{-t-\lambda(t)} \left(\zeta_0 + E\,q(0,t) + E \int_0^t e^{\tau}\,q(\tau,t)\,d\tau \right).$$

We further note, through integration by parts, that

$$\int_0^t e^{\tau}\, q(\tau, t)\, d\tau = -q(0, t) + \int_0^t e^{s+\lambda(s)}\, ds.$$

Thus we have

$$\int_{\mathbb{R}} v\, g^{(n)}(v, t)\, dv = \zeta_0\, e^{-t-\lambda(t)} + E\, e^{-t-\lambda(t)} \int_0^t e^{s+\lambda(s)}\, ds. \tag{44}$$

We are now left to show that the right hand side of (44) and $\bar{\zeta}(t)$ are equal. Towards this, we let

$$\hat{\zeta}(t) = \zeta_0\, e^{-t-\lambda(t)} + E\, e^{-t-\lambda(t)} \int_0^t e^{s+\lambda(s)}\, ds.$$

and note that, $\hat{\zeta}(t)$ fulfills

$$\frac{d}{dt}(e^t\, \hat{\zeta}(t)) = E\, e^t(1 - \bar{\zeta}(t)\, \hat{\zeta}(t)),$$

with $\hat{\zeta}(0) = \zeta_0$. Similarly, $\bar{\zeta}(t)$ satisfies

$$\frac{d}{dt}(e^t\, \bar{\zeta}(t)) = E\, e^t(1 - \bar{\zeta}(t)\, \bar{\zeta}(t)),$$

with $\bar{\zeta}(0) = \zeta_0$.

Since $\bar{\zeta}(t)$ is bounded we conclude that $\hat{\zeta}(t) = \bar{\zeta}(t)$. This together with (44) finally implies the result that

$$\int_{\mathbb{R}} v\, g^{(n)}(v, t)\, dv = \bar{\zeta}(t).$$

To prove (34) we first note

$$\begin{aligned} e^{\Lambda(t)} \int_{\mathbb{R}} v^2\, g^{(n)}\big(\psi_t(v), t\big)\, dv &= \int_{\mathbb{R}} v^2\, f_0(v)\, dv \\ &+ \int_{\mathbb{R}} \int_0^t e^{\Lambda(\tau)} v^2 Q_+\big(g^{(n-1)}, g^{(n-1)}\big)(\psi_\tau(v), \tau)\, d\tau\, dv. \end{aligned} \tag{45}$$

Use of (25), with the results from (32) and (33), makes the left hand side to be equal to

$$e^{t+2\lambda(t)}\, M_2^{(n)}(t) - 2\, E\, \bar{\zeta}(t)\, q(0, t)\, e^{t+\lambda(t)} + E^2\, q(0, t)^2\, e^t,$$

where $M_2^{(n)}(t) = \int_{\mathbb{R}} v^2 g^{(n)}(v, t)\, dv$.

Use of (25) and (26) with the remarks in (36)–(38) makes the second term in the sum at the right hand side of (45) to take the form

$$\int_0^t e^{\tau+2\lambda(\tau)}\, M_2^{(n-1)}(\tau)\, d\tau + E^2 \int_0^t e^{\tau}\, q(\tau)^2\, d\tau.$$

We put these pieces together, rearrange the terms to rewrite (45) as

$$e^{t+2\lambda(t)} M_2^{(n)}(t) = 2E\bar{\zeta}(t) q(0,t) e^{t+\lambda(t)} - E^2 q(0,t)^2 e^t + 1 + \int_0^t e^{\tau+2\lambda(\tau)} M_2^{(n-1)}(\tau)\, d\tau + E^2 \int_0^t e^\tau q(\tau)^2 d\tau. \quad (46)$$

Next we study the time derivatives of both sides of (46). But first we note

$$2E \frac{d}{dt}\left(\bar{\zeta}(t)\, q(0,t)\, e^{t+\lambda(t)}\right) = 2\,\lambda''(t)\, q(0,t)\, e^{t+\lambda(t)} + 2\left(\lambda'(t) + \lambda'(t)^2\right) \times q(0,t)\, e^{t+\lambda(t)} + 2\,\lambda'(t)\, e^{t+2\lambda(t)}.$$

We also have $\lambda''(t) = E^2 - \lambda'(t) - \lambda'(t)^2$, so that

$$2E \frac{d}{dt}\left(\bar{\zeta}(t)\, q(0,t)\, e^{t+\lambda(t)}\right) = 2\, E^2\, q(0,t)\, e^{t+\lambda(t)} + 2\,\lambda'(t)\, e^{t+2\lambda(t)}.$$

Since $\frac{d}{dt} q(0,t) = e^{\lambda(t)}$,

$$-E^2 \frac{d}{dt}\left(q(0,t)^2\, e^t\right) = -2E^2 q(0,t) e^{t+\lambda(t)} - E^2 q(0,t)^2 e^t.$$

Combining this with a trivial differentiation of the last two terms in (46) gives

$$\frac{d}{dt}\left(e^{t+2\lambda(t)} M_2^{(n)}(t)\right) = 2\,\lambda'(t)\, e^{t+2\lambda(t)} + e^{t+2\lambda(t)}\, M_2^{(n-1)}(t).$$

We rewrite the right hand side of this and get

$$\frac{d}{dt}\left(e^{t+2\lambda(t)}\, M_2^{(n)}(t)\right) = 2\,\lambda'(t)\, e^{t+2\lambda(t)} + e^{t+2\lambda(t)} + e^{t+2\lambda(t)}\left(M_2^{(n-1)}(t) - 1\right).$$

Supposing that $M_2^{(n-1)}(t) = 1$, then the last term in the above sum is zero so that the remaining terms can be rewritten as

$$\frac{d}{dt}\left(e^{t+2\lambda(t)}\, M_2^{(n)}(t)\right) = \frac{d}{dt}\left(e^{t+2\lambda(t)}\right).$$

Integration and rearrangement of the terms gives

$$M_2^{(n)}(t) = \left(M_2^{(n)}(0) - 1\right) e^{-t-2\lambda(t)} + 1.$$

This leads to the result that $M_2^{(n)}(t) = 1$, namely

$$\int_{\mathbb{R}} v^2\, g^{(n)}(v,t)\, dv = 1.$$

Hence we have proven that (32), (33), and (34) hold, and this concludes the proof of Proposition 1.

Remark 3. In very much the same way one can prove that all moments that initially are bounded remain bounded. In fact, one can find a closed system of ordinary differential equations for the moments $M_k(t), (k = 1, 2, \ldots)$, of the solutions to the thermostatted Kac equation:

$$\frac{d}{dt}M_k(t) = k\,E\,M_{k-1}(t) - k\,E\,\zeta(t)\,M_k(t) - M_k(t) + \int_{\mathbb{R}} v^k\,Q_+(f,f)(v)\,dv, \tag{47}$$

where the last term can be computed explicitly in terms of $M_j(t)$, $j \le k$.

In[3] we studied the stationary problem corresponding to (1), namely

$$E\,\frac{d}{dv}((1-\zeta\,v)\,f(v)) = Q(f,f)(v), \tag{48}$$

where $\zeta = \zeta_+$ is given by (17). The main result in[3] is

Theorem 2. *For all field strengths $E > 0$, the stationary problem (48) has a solution $f_\infty > 0$, that satisfies*

i) $\int_{\mathbb{R}} f_\infty(v)\,dv = 1,\ \int_{\mathbb{R}} v\,f_\infty(v)\,dv = \zeta$, *and* $\int_{\mathbb{R}} v^2\,f_\infty(v)dv = 1$.
ii) *Moments of any order of* f_∞ *are finite.*
iii) $f_\infty \in \mathcal{C}(\mathbb{R}\{\kappa\})$, *where* $\kappa = \frac{1}{\zeta}$.
iv) *For* $E < \sqrt{2}$, $f_\infty \in \mathcal{C}(\mathbb{R})$;
for $E = \sqrt{2}$, f_∞ *has a logarithmic singularity near* $v = \sqrt{2}$; *and*
for $E > \sqrt{2}$, f_∞ *has a singularity of the form* $|v-\kappa|^\gamma$ *near* $v = \kappa$, *where* $\gamma = \frac{1}{E\,\zeta} - 1$.

4. CONVERGENCE TO A STATIONARY STATE

The purpose of this section is to prove that the solutions to (1) converge to the stationary state as $t \to \infty$. For the Boltzmann equation, this is usually achieved by showing that the entropy is a decreasing functional attaining its infimum only at the equilibrium solution. In the current case, there is no obvious choice of an entropy, and so instead we will use a method introduced.[11,12] We recall the technique, which is based on the Fourier transform.

The Fourier transform of Eq. (1) without the force term, obtained by multiplication by $\exp(iv\xi)$ and integration over $\mathbb{R}$, is

$$\frac{\partial \hat{f}(\xi,t)}{\partial t} = \int_{-\pi}^{\pi} \big(\hat{f}(\xi\cos\theta,t)\hat{f}(\xi\sin\theta,t) - \hat{f}(\xi,t)\hat{f}(0,t)\big)\frac{d\theta}{2\pi}. \tag{49}$$

The mass is conserved (and set equal to one, $\hat{f}(0,t) \equiv 1$) and therefore (49) can be written as

$$\frac{\partial \hat{f}(\xi,t)}{\partial t} + \hat{f}(\xi,t) = \widehat{Q}_+(\hat{f},\hat{f})(\xi), \tag{50}$$

where

$$\widehat{Q}_+(\hat{f},\hat{g})(\xi) = \int_{-\pi}^{\pi} \hat{f}(\xi\cos\theta,t)\hat{g}(\xi\sin\theta,t)\frac{d\theta}{2\pi}.$$

Also energy is conserved (equal to one), so $-\hat{f}''(0,t) = \int_{\mathbb{R}} v^2 f(v,t)\,dv \equiv 1$[3] and if in addition $\int_{\mathbb{R}} f(v,0)\,dv = 0$, then it remains zero, and so $\hat{f}'(0,t) \equiv 0$.

If f and g are two such solutions, their difference satisfies $\hat{f}(\xi,t) - \hat{g}(\xi,t) = o(\xi^2)$, and if moments of order three of f and g are both bounded then for all $p \in [0,1)$,

$$w(\xi,t) = \frac{\hat{f}(\xi,t) - \hat{g}(\xi,t)}{|\xi|^{2+p}}$$

is bounded for all times, and

$$\frac{\partial w(\xi,t)}{\partial t} + w(\xi,t) = \int_{-\pi}^{\pi} (w(\xi\cos\theta,t)|\cos\theta|^{2+p}\hat{f}(\xi\sin\theta,t) + w(\xi\sin\theta,t) \\ |\sin\theta|^{2+p}\hat{g}(\xi\cos\theta,t))\frac{d\theta}{2\pi}. \tag{51}$$

From this we can deduce that for any $\mathrm{X} > 0$,

$$\sup_{|\xi|<\mathrm{X}} \left| \frac{\partial w(\xi,t)}{\partial t} + w(\xi,t) \right| \le \sup_{|\xi|<\mathrm{X}} |w(\xi,t)| \int_{-\pi}^{\pi} \left(|\cos\theta|^{2+p} + |\sin\theta|^{2+p}\right)\frac{d\theta}{2\pi}$$

and it follows that

$$\sup_{|\xi|<\mathrm{X}} |w(\xi,t)| \le \sup_{|\xi|<\mathrm{X}} |w(\xi,0)|\, e^{-(1-\alpha)t},$$

where $\alpha = \int_{-\pi}^{\pi}(|\cos\theta|^{2+p} + |\sin\theta|^{2+p})\frac{d\theta}{2\pi} < 1$. Thus $\hat{f} - \hat{g}$ converges pointwise to zero, exponentially in time. In the particular case where g is the Maxwellian with correct moments, this proves that the probability density f converges to equilibrium *in distribution*.

When the force term is added to (49) we cannot obtain exactly the same result, because $i\hat{f}'(0,t) = \zeta_f(t) = \int_{\mathbb{R}} f(v,t)v\,dv$, which depends on time. (Note the

[3] In this section, the symbol $'$, always denotes a derivative with respect to ξ.

subscript f on ζ, which is important when two different functions are considered). Instead we will now study

$$u(\xi, t) = \hat{f}(\xi, t) + i(\zeta_f(t) - \zeta_g(t))\,\phi(\xi) - \hat{g}(\xi, t). \tag{52}$$

where $\phi(\xi)$ is a smooth bounded function that satisfies $\phi(\xi) = \xi$ for $|\xi| \le 1$. The Fourier coefficients of u up to order two vanish, so $u(\xi, t)/|u|^{2+p}$ is well-defined for all t, if $0 \le p < 1$.

The Fourier transform of Eq. (1) is

$$\frac{\partial \hat{f}(\xi, t)}{\partial t} + iE\xi \hat{f}(\xi, t) + E\zeta_f(t)\xi \hat{f}'(\xi, t) + \hat{f}(\xi, t) = \widehat{Q}_+(\hat{f}, \hat{f})(\xi, t).$$

(More correctly, one could have written $\widehat{Q}_+(\hat{f}(\cdot, t), \hat{f}(\cdot, t))(\xi)$ in the right hand side.) Then

$$\frac{\partial u}{\partial t} + u = i\phi(\xi)\left(\frac{d\zeta_f}{dt} - \frac{d\zeta_g}{dt}\right) + i\phi(\xi)(\zeta_f - \zeta_g) - iE\xi\hat{f} - E\zeta_f\xi\hat{f}'$$
$$+ iE\xi\hat{g} + E\zeta_g\xi\hat{g}' + \widehat{Q}_+(\hat{f} - \hat{g}, \hat{f}) + \widehat{Q}_+(\hat{g}, \hat{f} - \hat{g})$$

After rearranging the terms, we find

$$\frac{\partial u}{\partial t} + u = i\left(\frac{d\zeta_f}{dt} - \frac{d\zeta_g}{dt}\right)\phi - E\xi(\zeta_f - \zeta_g)\hat{f}' - E\xi\zeta_g u$$
$$+ Ei\xi\zeta_g(\zeta_f - \zeta_g)\phi' - iE\xi u - E\xi(\zeta_f - \zeta_g)\phi + \widehat{Q}_+(u, \hat{f})$$
$$- i(\zeta_f - \zeta_g)\widehat{Q}_+(\phi, \hat{f}) + \widehat{Q}_+(\hat{g}, u) - i(\zeta_f - \zeta_g)\widehat{Q}_+(\hat{g}, \phi).$$

Next we note that

$$E\,\xi(\zeta_f - \zeta_g)\hat{f}' = E\xi(\zeta_f - \zeta_g)(\hat{f}' + i\zeta_f\phi' + \phi)$$
$$+ i\,E\,\xi\,\zeta_f(\zeta_f - \zeta_g)\phi' + E\,\xi(\zeta_f - \zeta_g)\phi,$$

and that the first of these terms is $\mathcal{O}(|\xi|^3)$ near $\xi = 0$, and that it is bounded by $C(1 + |\xi|)$. Moreover, $\frac{d}{dt}\zeta_f = E(1 - \zeta_f^2) - \zeta_f$. A new rearrangement of the terms now gives

$$\frac{\partial u}{\partial t} + u + iE\xi u + E\xi\zeta_g u = \widehat{Q}_+(u, \hat{f}) + \widehat{Q}_+(\hat{g}, u) + E\,i(\zeta_f^2 - \zeta_g^2)$$
$$\times (\xi\phi' - \phi) - E\xi(\zeta_f - \zeta_g)(\hat{f}' + i\zeta_f\phi' + \phi)$$
$$- i(\zeta_f - \zeta_g)(\widehat{Q}_+(\phi, \hat{f}) + \widehat{Q}_+(\hat{g}, \phi))$$

For $|\xi| \le 1$, $\phi(\xi) = \xi$, and so in this interval

$$\begin{aligned}\widehat{Q}_+(\phi, \hat{f}) &= \int_{-\pi}^{\pi} \phi(\xi)\cos(\theta)\hat{f}(\xi\sin(\theta))\frac{d\theta}{2\pi}\\ &= \phi(\xi)\int_{-\pi}^{\pi}\cos(\theta)(\hat{f}(\xi\sin(\theta)) - 1 + i\zeta_f\phi(\xi)\sin(\theta))\frac{d\theta}{2\pi}\\ &\quad + \phi(\xi)^2\int_{-\pi}^{\pi}\cos(\theta)(1 - i\zeta_f\phi(\xi)\sin(\theta))\frac{d\theta}{2\pi}.\end{aligned}$$

The second of these integrals vanish, and the first one is $\mathcal{O}(|\xi|^3)$ for small ξ. For large ξ, the expression is bounded. A similar calculation can be done for $\widehat{Q}_+(\hat{g}, \phi)$, and in summary we obtain

$$\begin{aligned}\frac{\partial u}{\partial t} + u + iE\xi u + E\xi\zeta_g u = \widehat{Q}_+(u, \hat{f}) + \widehat{Q}_+(\hat{g}, u) + (\zeta_f - \zeta_g)\\ \times (E\xi R_1(\xi, t) + R_2(\xi, t)),\end{aligned}$$

where $|R_1(\xi, t)| \le \min\left(C, |\xi|^2\right)$ and $|R_2(\xi, t)| \le \min\left(C, |\xi|^3\right)$ for some constant C.

Although the calculations can be carried out in exactly the same way for a general solution $\hat{g}$, we now concentrate on the case where $\hat{g}$ is the Fourier transform of the stationary solution to (1). Then $\zeta_g = \zeta_+ = (1 + \sqrt{1 + 4E^2})/2E$, and it follows directly from (16) that $\zeta_f(t) - \zeta_g$ s bounded by $C\exp(-\sqrt{1 + 4E^2}t)$.

For any function $w(\xi, t)$, we now define $w^\sharp(\xi, t) = w\left(e^{E\zeta_+ t}\xi, t\right)$. Expressed in this way,

$$\begin{aligned}\frac{\partial u^\sharp}{\partial t} + u^\sharp + Ei\xi e^{E\zeta_+ t} = \widehat{Q}_+(u^\sharp, \hat{f}^\sharp) + \widehat{Q}_+(\hat{g}^\sharp, u^\sharp) + (\zeta_f - \zeta_+)\\ \times (E\xi e^{E\zeta_+ t}R_1^\sharp + R_2^\sharp).\end{aligned}$$

We have used the fact that $\widehat{Q}_+(u, v)^\sharp = \widehat{Q}_+(u^\sharp, v^\sharp)$, which is easily verified.

Next with $v(\xi, t) = u^\sharp(\xi, t)\,e^t$,

$$\frac{\partial v}{\partial t} + Ei\xi e^{E\zeta_+ t}v = \widehat{Q}_+(v, \hat{f}) + \widehat{Q}_+(\hat{g}, v) + e^t(\zeta_f - \zeta_+)(E\xi e^{E\zeta_+ t}R_1^\sharp + R_2^\sharp),$$

from which it follows that

$$\frac{\partial|v|}{\partial t} \le \widehat{Q}_+\left(|v|, |\hat{f}|\right) + \widehat{Q}_+(|\hat{g}|, |v|) + e^t|(\zeta_f - \zeta_+)(E\xi e^{E\zeta_+ t}R_1^\sharp + R_2^\sharp)|,$$

or, using that $|\hat{f}| \le 1$ and $|\hat{g}| \le 1$,

$$\begin{aligned}|v(\xi, t)| \le |v(\xi, 0)| + \int_0^t\int_{-\pi}^{\pi}(|v(\xi\cos\theta, \tau)| + |v(\xi\cos\theta, \tau)|)\frac{d\theta}{2\pi}\,d\tau\\ + \int_0^t e^\tau|(\zeta_f(\tau) - \zeta_+)(E\xi e^{E\zeta_+,\tau}R_1^\sharp(\xi\tau) + R_2^\sharp(\xi, \tau))|\,d\tau. \qquad (53)\end{aligned}$$

The next step is to divide this expression by $|\xi|^{2+p}$, and to take the supremum over $|\xi| \leq \mathrm{X}$ just like in the case with no force term. To estimate the source term, recall that R_1 and R_2 are bounded and of the order $|\xi|^2$ and $|\xi|^3$, respectively, when ξ is small. Hence, for any $\mathrm{X} > 0$,

$$\sup_{|\xi|\leq \mathrm{X}} \frac{|R_1^{\sharp}(\xi,t)|}{|\xi|^{1+p}} = \sup_{|\xi|\leq \mathrm{X}} \frac{|R_1(\xi e^{E\zeta_+ t},t)|}{|\xi|^{1+p}} \leq C\, e^{(1+p)E\zeta_+ t},$$

and

$$\sup_{|\xi|\leq \mathrm{X}} \frac{|R_2^{\sharp}(\xi,t)|}{|\xi|^{2+p}} \leq C\, e^{(2+p)E\zeta_+ t},$$

where the constants are independent of X. Estimating the first integral on the right hand side of Eq. (53) in the same way as the estimate of $\widehat{Q}_+$ in (51) we get

$$\sup_{|\xi|\leq \mathrm{X}} \frac{|v(\xi,t|}{|\xi|^{2+p}} \leq \sup_{|\xi|\leq \mathrm{X}} \frac{|v(\xi,0|}{|\xi|^{2+p}} + \alpha \int_0^t \sup_{|\xi|\leq \mathrm{X}} \frac{|v(\xi,\tau|}{|\xi|^{2+p}}\, d\tau$$
$$+ \int_0^t e^{\tau} e^{(2+p)E\zeta_+ \tau} |(\zeta_f(\tau) - \zeta_+)|\, d\tau,$$

where $\alpha = \int_{-\pi}^{\pi} (|\sin\theta|^{2+p} + |\cos\theta|^{2+p}) \frac{d\theta}{2\pi}$. Because $|(\zeta_f(\tau) - \zeta_+)| \leq Ce^{-\sqrt{1+4E^2}t}$, the second integral is bounded by $C(E,p)\, e^{(1+(2+p)E\zeta_+ - \sqrt{1+4E^2})t}$, and using the Gronwall inequality,

$$\sup_{|\xi|\leq \mathrm{X}} \frac{|v(\xi,t)|}{|\xi|^{2+p}} \leq \sup_{|\xi|\leq \mathrm{X}} \frac{|v(\xi,0)|}{|\xi|^{2+p}} e^{\alpha t} + C(E,p) e^{\left(1+(2+p)E\zeta_+ - \sqrt{1+4E^2}\right)t}.$$

This holds for all X, and hence, with $v(\xi,t) = e^t u(\xi\, e^{E\zeta_+ t}, t)$,

$$\sup_{|\xi|} \frac{|u(\xi e^{E\zeta_+ t},t)|}{|\xi|^{2+p}} \leq \sup_{|\xi|} \frac{|u(\xi,0)|}{|\xi|^{2+p}} e^{(\alpha-1)t} + C(E,p) e^{\left((2+p)E\zeta_+ - \sqrt{1+4E^2}\right)t}.$$

Then, for all ξ, and all $t \geq 0$,

$$|u(\xi e^{E\zeta_+ t},t)| \leq |\xi|^{2+p}\left(C_0\, e^{(\alpha-1)t} + C(E,p)\, e^{\left((2+p)E\zeta_+ - \sqrt{1+4E^2}\right)t}\right),$$

or, which is the same,

$$|u(\xi,t)| \leq |\xi|^{2+p}\left(C_0 e^{(\alpha-1-(2+p)E\zeta_+)t} + C(E,p) e^{-\sqrt{1+4E^2}t}\right),$$

where C_0 depends only on the initial data. Now we recall that $u(\xi,t) = \hat{f}(\xi,t) - \hat{g}(\xi) + i\phi(\xi)(\zeta_f(\tau) - \zeta_+)$, where $\hat{g}$ is the Fourier transform of the solution to the stationary equation, and therefore

$$|\hat{f}(\xi,t) - \hat{g}(\xi)| \leq |\xi|^{2+p}\left(C_0 e^{(\alpha-1-(2+p)E\zeta_+)t} + C(E,p) e^{-\sqrt{1+4E^2}t}\right)$$
$$+\phi(\xi)\left|\zeta_f(t) - \zeta_+\right|.$$

This implies that $|\hat{f}(\xi,t)-\hat{g}(\xi)| \to 0$ point-wise, exponentially fast, and also that

$$\sup_{|\xi|\le X} \frac{|\hat{f}(\xi,t)-\hat{g}(\xi)|}{|\xi|} \;\le\; C\, X^{1+p} e^{-\sqrt{1+4E^2}t}.$$

In conclusion we have proven the following theorem:

Theorem 3. *Let $\hat{f}(\xi,t)$ be the Fourier transform of the solution to Eq. (1), and let $\hat{g}(\xi)$ be the Fourier transform of the solution to the stationary problem. Then*

$$|\hat{f}(\xi,t)-\hat{g}(\xi)| \le (|\xi|^{2+p}+|\xi|)e^{-\sqrt{1+4E^2}t},$$

which implies that $f(v,t)$ converges in distribution to the stationary state

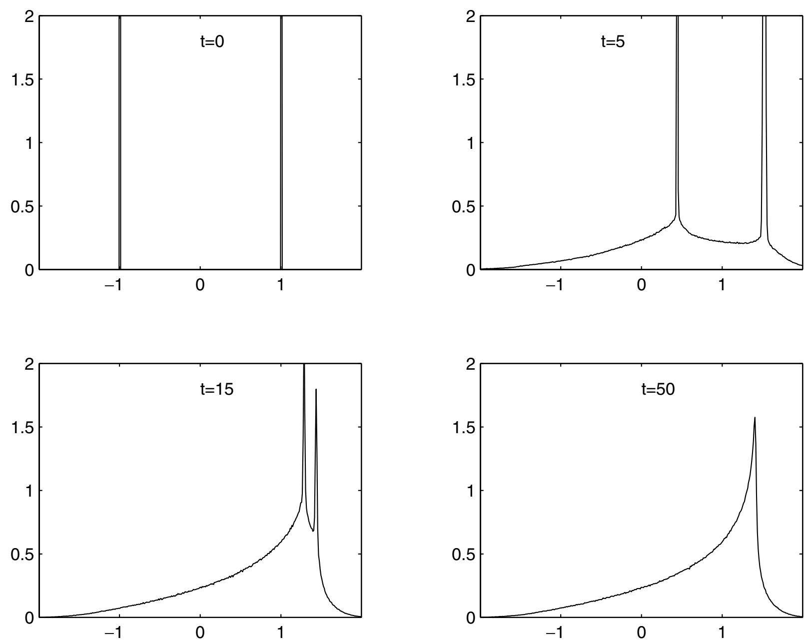

Fig. 1. Time evolution of $f(v,t)$, the solution to the thermostatted Kac equation. (simulated with the Monte Carlo method using $N = 5000$ particles, $E = \sqrt{2}$.)

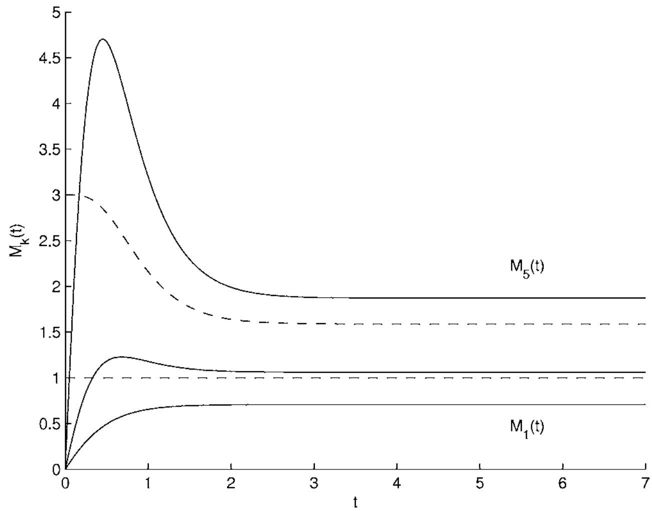

Fig. 2. Time evolution of the first five moments of f according to (47). ($E = \sqrt{2}$).

5. SIMULATION RESULTS

In this section we present some numerical approximations of the solutions to the thermostatted Kac equation using the Monte Carlo method. This corresponds to simulating a large number of trajectories to the jump processes defined on $\mathbb{S}^{N-1}(\sqrt{N})$, where N is the number of particles. These trajectories can be computed exactly thanks to the fact that there is an explicit solution to the evolution of $\mathbf{V}$ in between the jumps.

For sufficiently large N we expect the one-particle marginal $f_1^N(v, t)$ to approach $f(v, t)$–the solution to (48). Figure 1 shows the time evolution starting from an initial data consisting of two Dirac masses. The number of particles, N was 5000, and the force field had strength $E = \sqrt{2}$. As time evolves, the density approaches the stationary state, which according to the study,[3] has a logarithmic singularity in this case.

Having a closed system of ordinary differential equations for $M_k(t) = \int_{\mathbb{R}} v^k f(v, t)\, dv$ we can explicitly compute the first few. Figure 2 shows the time evolution of the first five moments of the solution according to (47) with $E = \sqrt{2}$.

For the three dimensional Boltzmann equation, the density converges to a Dirac mass when $t \to \infty$. This is illustrated in the last example, where the force field is $\mathbf{E} = (0.1, 0, 0)$, and the simulation is carried out with 5000 particles. The initial distribution of the velocities is taken to be

$$f_0(v_1, v_2, v_3) = \frac{1}{2}(\delta_{-1}(v_1) + \delta_{+1}(v_1))\, \frac{1}{2\pi} \exp\left(- v_2^2 - v_3^2 \right).$$

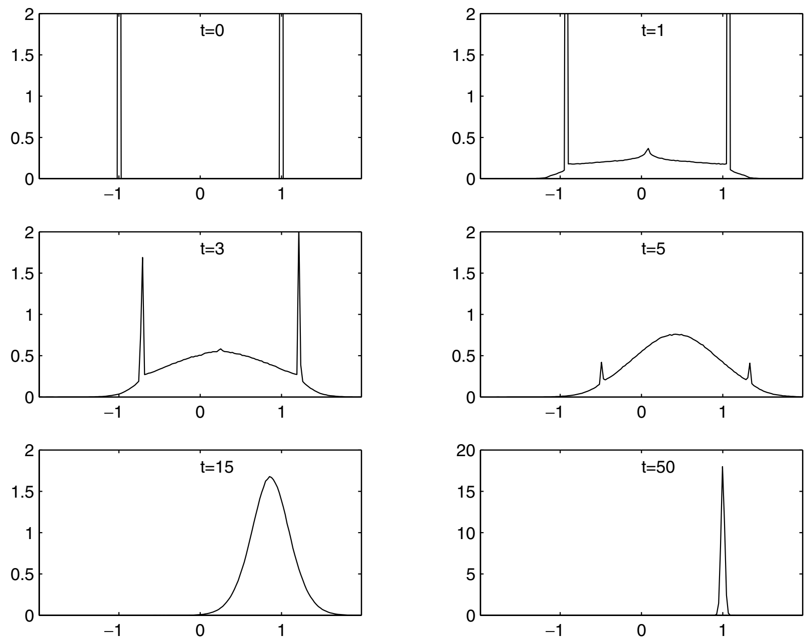

Fig. 3. Time evolution of the v_1-marginal of $f(\mathbf{v}, t)$ of the solution to the thermostatted Boltzmann equation. (simulated with 5000 particles). In fact, with $\alpha(t)$ and $\beta(t)$ suitably chosen, $g(\mathbf{v}, t) = f((\mathbf{v} - \zeta(t))/\alpha(t), \beta(t))$ satisfies the usual spatially homogeneous Bolzmann equation, which explains the apparent relaxation to a peaked Maxwellian.

The v_1-marginal, $\int f(v_1, v_2, v_3, t)\, dv_2 dv_3$ is shown in Fig. 3. The other marginals converge to a Dirac measure at zero.

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